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Two Results Concerning Asymptotic Behavior of Solutions of the Burgers Equation with Force

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We consider the Burgers equation with an external force. For the case of the force periodic in space and time we prove the existence of a solution periodic in space and time which is the limit of a wide class of solutions as $t \to \infty$. If the force is the product of a periodic function of x and white noise in time, we prove the existence of an invariant distribution concentrated on the space of space-periodic functions which is the limit of a wide class of distributions as $t \to \infty$.

KEY WORDS: Burgers equation; white noise; local central limit theorem; partition function.

1. FORMULATION OF THE RESULTS

We consider the one-dimensional Burgers equation with force having the form

$$u_t + u \cdot u_x = \mu u_{xx} + F'(x)\, B(t), \qquad -\infty < x < \infty \tag{1}$$

Here $F(x)$ is a C^1-periodic function of period x_0. The assumptions concerning B will be formulated later. The initial data $u(x;0)$ are derivatives $u(x;0) = v'(x)$, where $v(x)$ are typical realizations of some random process. The probability distribution corresponding to v is denoted by P_0. It is defined on the natural σ-algebra of subsets of the space V of absolutely continuous functions $v(x)$. We assume that:

1. There exists a constant C_0 such that with P_0-probability 1

$$|v(x)| \leq C_0$$

for all x.

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2. $A = E\{\exp[-v(x)]\}$ does not depend on x.
3. There exists γ, $0 < \gamma < 1/2$, such that for P_0-almost every $v(x)$,

$$\lim_{n \to \infty} \sup_{\substack{a \in [0, x_0] \\ |m| \leqslant n, m \in \mathbb{Z}^1}} \frac{1}{2[n^\gamma] + 1} \left| \sum_{|k-m| \leqslant [n^\gamma]} e^{-v(a + kx_0)} - A \right| = 0$$

It is easy to give concrete examples of P_0 for which condition 3 is true.

Theorem 1. Let B be a continuous periodic function of period τ_0. There exists a solution $u^{(0)}(x, t)$ of (1) periodic in x with period x_0 and periodic in time with period τ_0 such that for P_0-a.e. v

$$\lim_{t \to \infty} [u(x, t) - u^{(0)}(x, t)] = 0$$

for any x, $-\infty < x < \infty$.

Remarks. 1. Our method of proof also gives an explicit expression for $u^{(0)}(x, t)$.

2. The theorem remains true if the force in the Burgers equation takes the form $\partial F(x, t)/dx$, where $F(x, t)$ is a function periodic in space with period x_0 and periodic in time with period τ_0.

3. The theorem remains true for bounded functions v such that $v'(x) \to 0$ as $x \to \pm\infty$, $v(x) \to \text{const}$ as $x \to \pm\infty$.

4. The convergence in Theorem 1 is pointwise. After giving the proof, we discuss stronger statements concerning the character of convergence.

In Theorem 2 we assume that $B(t)$ is a white noise. This means that for any t_1, t_2, $t_1 < t_2$, a random variable $b(t_1, t_2) = \int_{t_1}^{t_2} B(\tau)\, d\tau$ is defined such that:

(a_1) $b(t_1, t_2)$ has the Gaussian distribution with mean value equal to zero and dispersion $Eb^2(t_1, t_2) = \sigma(t_2 - t_1)$ for some $\sigma > 0$.

(a_2) For nonoverlapping intervals (t_1', t_2') and (t_1'', t_2'') the random variables $b(t_1', t_2')$ and $b(t_1'', t_2'')$ are independent.

Denote by $M((t_1, t_2)$ the least σ-algebra generated by all $b(t_1', t_2')$, where $t_1 < t_1' < t_2' < t_2$, and let $\{T^t\}$ be the measure-preserving flow on the space of all random variables $b(t_1', t_2')$, where each T^t transforms $M(t_1, t_2)$ to $M(t_1 - t, t_2 - t)$ and

$$(T^t b)(t_1', t_2') = b(t_1 - t', t_2 - t')$$

for any t_1', t_2', $t_1 + t < t_1' < t_2' < t_2 + t$.

Assume that P_0 satisfies the same conditions as in Theorem 1.

Theorem 2. Let P_t be the natural probability distribution on the space of solutions $u(x, t)$ of (1) induced by P_0, $0<t<\infty$. Then P_t converges weakly as $t \to \infty$ to some probability distribution Q which does not depend on P_0 and is concentrated on the space of functions periodic in x with period x_0.

The proof of Theorem 1 is given in Section 2. In Section 3 we expound the proof of Theorem 2. The actual statement which we show is the following. For any $\varepsilon<0$ we find $t_0(\varepsilon)$, a set $C \in M(0, t_0(\varepsilon))$, and a functional $H_x(\{B(t_1, t_2)\}, 0 \leqslant t_1, t_2 \leqslant t_0(\varepsilon))$ defined on C and such that $\text{Prob}(C) \geqslant 1-\varepsilon$ and if $T^{-t}b \in C$, then

$$|u(x, t) - H_x(b(t_1, t_2), t-t_0(\varepsilon) \leqslant t_1, t_2 \leqslant t)| \leqslant \varepsilon$$

In other words, for increasing t, the solution $u(x, t)$ becomes a functional of the realization of white noise $B(\tau)$, $0<\tau<t$, with "short memory." This memory can be estimated in a more precise way. The functional H_x depends periodically on x. Theorems 1 and 2 are valid also for the multidimensional Burgers equation. Only small modifications in the proofs are needed.

2. PROOF OF THEOREM 1

After the appropriate rescaling of x and μ we may assume that the period $\tau_0 = 1$. We use the Cole–Hopf substitution $u = -2\mu(\varphi_x/\varphi)$,[1,2] and get for φ the equation

$$\varphi_t = \mu\varphi_{xx} - \frac{1}{2\mu} F(x)\, B(t)\, \varphi \tag{2}$$

The Feynman–Kac formula[3] makes it possible to write down φ as a functional integral. Namely, denote by $\Pi^{(t_1, t_2)}_{W_1, W_2}$ the corresponding Wiener measure on the space of continuous functions $w(\tau)$, $t_2 \leqslant \tau \leqslant t_1$, such that $w(t_1) = w_1$, $w(t_2) = w_2$. Then

$$\varphi(x; t) = \int_{-\infty}^{\infty} dy\, \{\exp[-v(y)]\} \times \int \left\{\exp\left[\int_0^t F(W(\tau))\, B(\tau)\, d\tau\right]\right\} d\Pi^{(t,0)}_{x,y}(W) \tag{3}$$

Put $t = t_0$, $t - j = t_j$, $j \geqslant 1$, and $j \in \mathbb{Z}^1$, and find r such that $t_{r+2} < 0 < t_{r+1}$. Fix the numbers $a_1, a_2, \ldots, a_r, a_{r+1}$, $a_j \in [0, x_0)$ and rewrite (3) as follows:

$$\varphi(x;t)=\int_0^{x_0}\cdots\int_0^{x_0} da_1\,da_2\cdots da_r\,da_{r+1}\sum_{\substack{m_1,m_2,\ldots,m_{r+1}\\ m_j\in\mathbb{Z}^1}} K_{t_1}(x,a_1+mx_0)$$

$$\times\prod_{j=2}^{r} K_{t_j}(a_{j-1}+m_jx_0,a_j+m_jx_0)$$

$$\times K_f(a_r+m_rx_0,a_{r+1}+m_{r+1}x_0)\,e^{-v(a_{r+1}+m_{r+1}x_0)} \tag{4}$$

Here

$$K_{t_j}(W_1,W_2)=\int\left\{\exp\left[\int_{t_{j-1}}^{t_j} F(W(\tau))\,B(\tau)\,dt\right]\right\}d\Pi^{(t_j,t_{j-1})}_{(W_1,W_2)}(W)$$

$$K_f(W_1,W_2)=\int\left\{\exp\left[\int_0^{t_r} F(W(\tau))\,B(\tau)\,dt\right]\right\}d\Pi^{(t_r,0)}_{(W_1,W_2)}(W)$$

The periodicity of F in space and that of B in time imply the following relations:

1. $K_{t_j}(W_1,W_2)=K_{t_j}(W_1+mx_0,W_2+mx_0),\qquad 2\leqslant j\leqslant r$

for all $m\in\mathbb{Z}^1$.

2. $K_{t_2}(W_1,W_2)=K_{t_3}(W_1,W_2)=\cdots K_{t_r}(W_1,W_2)=K(W_1,W_2)$

The functions $K_{t_1}(w_1,w_2)$, $K_f(w_1,w_2)$ depend on the fractional part $\{t\}$ and thus are periodic in time with period 1. Introduce the sums

$$Z_{t_1}(x;a_1)=\sum_{m\in\mathbb{Z}^1} K_{t_1}(x,a_1+mx_0)$$

$$Z_{t_j}(a_{j-1},a_j)=Z_{t_j}(a_{j-1},a_j)$$

$$=\sum_{m\in\mathbb{Z}^1} K_{t_j}(a_{j-1},a_j+mx_0),\qquad 2\leqslant j\leqslant r$$

$$Z_f(a_r,a_{r+1})=\sum_{m\in\mathbb{Z}^1} K_f(a_r,a_{r+1}+mx_0)$$

and the probabilities

$$p_{t_1}(x,a_1+m_1x_0)$$
$$=Z_{t_1}^{-1}(x,a_1)\,K_{t_1}(x,a_1+m_1x_0)$$

$$p_{t_j}(a_{j-1}+m_{j-1}x_0,a_j+m_jx_0)$$
$$=Z_{t_j}^{-1}(a_{j-1},a_j)\,K_{t_j}(a_{j-1}+m_{j-1}x_0,a_j+m_jx_0),\qquad 2\leqslant j\leqslant r$$

$$p_f(a_r+m_rx_0;a_{r+1}+m_{r+1}x_0)$$
$$=Z_f^{-1}(a_r,a_{r+1})\,K_f(a_r+m_rx_0;a_{r+1}+m_{r+1}x_0)$$

Consider the sequence of independent random variables ξ_1, $\xi_2, ..., \xi_{r+1}$, where each variable ξ_j takes values $m \in \mathbb{Z}^1$, and ξ_1 has the distribution with the probabilities $p_1(x, a_1 + mx_0)$, while ξ_j, $2 \leqslant j \leqslant r$, have the distribution with the probabilities $p_{t_j}(a_{j-1}, a_j + mx_0) = p(a_{j-1}, a_j + mx_0)$, which depends only on $\{t\}$ but not on j, and ξ_{r+1} has the distribution with the probabilities $p_f(a_r, a_{r+1} + mx_0)$, which also depend only on $\{t\}$. Then the sum in (4) can be rewritten as follows:

$$\begin{aligned}
\sum &= \sum (a_1, a_2, ..., a_{r+1}) \\
&= \sum_{m_1, m_2, ..., m_{r+1}} K_{t_1}(x, a_1 + m_1 x_0) \\
&\quad \times \prod_{j=2}^{r} K_{t_1}(a_{j-1} + m_{j-1} x_0, a_j + m_j x_0) \\
&\quad \times K_f(a_r + m_r x_0, a_{r+1} + m_{r+1} x_0)\, e^{-v(a_{r+1} + m_{r+1} x_0)} \\
&= Z_{t_1}(x; a_1)\, Z(a_1, a_2) \cdots Z(a_{r-1}, a_r)\, Z_f(a_r, a_{r+1}) \\
&\quad \times \sum_{m_1, m_2, ..., m_{r+1}} p_{t_1}(x, a_1 + m_1 x_0) \\
&\quad \times \prod_{j=2}^{r} p_j(a_{j-1} + m_{j-1} x_0, a_j + m_j x_0) \\
&\quad \times p_f(a_r + m_r x_0, a_{r+1} + m_{r+1} x_0)\, e^{-v(a_{r+1} + m_{r+1} x_0)} \\
&= Z_{t_1}(x; a_1) \prod_{j=2}^{r} Z_j(a_{j-1}, a_j)\, Z_f(a_r, a_{r+1}) \\
&\quad \times \sum_{n_1, n_{r+1}} p_{t_1}(x, a_1 + n_1 x_0) \prod_{j=2}^{r} p_j(a_{j-1}, a_j + n_j x_0) \\
&\quad \times p_f(a_r, a_{r+1} + n_{r+1} x_0)\, e^{-v(a_{r+1} + (n_1 + \cdots + n_{r+1})\, x_0)} \\
&= Z_{t_1}(x; a_1) \prod_{j=2}^{r} Z(a_{j-1}, a_j)\, Z_f(a_r, a_{r+1}) \\
&\quad \times E_\xi e^{-v(a_{r+1} + (\xi_1 + \cdots + \xi_{r+1})\, x_0)}
\end{aligned} \tag{5}$$

where E_ξ is the expectation with respect to the joint distribution of the random variables ξ_j, $1 \leqslant j \leqslant r+1$. Put $\mu(a_1) = E\xi_1$, $\mu(a_{j-1}, a_j) = E\xi_j$ for $1 \leqslant j \leqslant r+1$, $d(a_1) = D(\xi_1) = E_\xi(\xi_1 - \mu(a_1))^2$, $d(a_{j-1}, a_j) = E_\xi(\xi_j - \mu(a_{j-1}, a_j))^2$, $M = \mu(a_1) + \sum_{j=2}^{r+1} \mu(a_{j-1}, a_j)$, and $D = d(a_1) + \sum_{j=2}^{r+1} d(a_{j-1}, a_j)$.

Lemma 1. Under the conditions of Theorem 1, the sequence of random variables $\xi_1, \xi_2, ..., \xi_{r+1}$ satisfies the central local limit theorem of probability theory in the form

$$\sup_{a_1, ..., a_{r+1}} \left| P_\xi(\xi_1 + \cdots + \xi_{r+1} = m) - \frac{1}{(2\pi D)^{1/2}} e^{-(m-M)^2/2D} \right| \leqslant \varepsilon_t \tag{6}$$

where ε_t tends to zero as $t \to \infty$.

The statement of Lemma 1 means that the convergence to zero of the difference in (6) is uniform in m_1 and $a_1, a_2, ..., a_{r+1}$. Lemma 1 can be easily proven by standard methods of probability theory (see, e.g., ref. 4). We omit the proof. During the proof one must keep in mind the boundedness of $|F|$ and $|B|$.

Consider in more detail the expectation

$$E = E_\xi e^{-v_0(a_{r+1} + (\xi_1 + \cdots + \xi_{r+1}) x_0)}$$

In view of Lemma 1 and property 1 of P_0, it is equal to

$$E = \sum_m e^{-v_0(a_{r+1} + m x_0)} P_\xi\{\xi_1 + \cdots + \xi_{r+1} = m\}$$

$$= \sum_m e^{-v_0(a_{r+1} + m x_0)} \frac{1}{(2\pi D)^{1/2}} e^{-(m-M)^2/2D} + e^{c_0} \delta_t$$

where $\delta_t \to 0$ as $t \to \infty$. Using property 3 of P_0, we easily get

$$E = A + \delta_t^{(1)}$$

where $\delta_t^{(1)} \to 0$ as $t \to \infty$ uniformly in $a_1, ..., a_{r+1}$. Thus,

$$\varphi(x; t) \sim A \int \cdots \int da_1\, da_2 \cdots da_r\, da_{r+1}\, Z_{t_1}(x; a_1)$$

$$\times \prod_{j=2}^{r} Z(a_{j-1}, a_j)\, Z_f(a_r, a_{r+1}) \tag{7}$$

The expression (7) can be studied with the methods of statistical mechanics. Consider $Z(a', a'')$ as a transfer matrix of a one-dimensional system and find its positive eigenvector $e(a)$ and the corresponding positive eigenvalue λ:

$$\int e(a')\, Z(a', a'')\, da' = \lambda e(a'')$$

Introduce the Markov transition operator π with the transition probabilities

$$\pi(a', a'') = \frac{Z(a', a'')\, e(a')}{\lambda e(a'')}$$

giving the density of the transition $a'' \to a'$. Then (7) can be rewritten as

$$\begin{aligned}\varphi(x; t) &\sim A\lambda^r \int \cdots \int da_1 \cdots da_r\, da_{r+1}\, Z_{t_1}(x; a_1)[e(a_1)]^{-1}\, \pi(a_2, a_3)\\ &\quad \times \cdots \pi(a_{r-1}, a_r)\, Z(a_r, a_{r+1})\, e(a_r)\\ &= A\lambda^r \int Z_{t_1}(x; a_1)[e(a_1)]^{-1}\, \pi^{(r)}(a_1, a_r)\\ &\quad \times Z(a_r, a_{r+1})\, e(a_r)\, da_1\, da_r\, da_{r+1}\end{aligned}$$

The ergodic theorem for the Markov chain generated by the operator π shows that $\pi^{(r)}(a_1, a_r)$ asymptotically does not depend on a_r and is exponentially close to the stationary distribution of this chain. Denote this distribution by $\pi(a_1)$. It is well known that it has the form $e(a_1)\, e^*(a_1)$, where $e^*(a_1)$ is the positive eigenvector of the adjoint operator, i.e.,

$$\int Z(a', a'')\, e^*(a'')\, da'' = \lambda e^*(a')$$

Thus

$$\varphi(x, t) \sim A \cdot A_1 \lambda^r \int Z_{t_1}(x; a_1)\, e^*(a_1)\, da_1$$

where $A_1 = \iint Z(a_r, a_{r+1})\, e(a_r)\, da_r\, da_{r+1}$.

Taking the derivative of the rhs of (4) with respect to x and making the same analysis, we find a similar expression for φ_x:

$$\varphi_x(x, t) \sim AA_1 \lambda^r \int \frac{\partial}{\partial x} Z_{t_1}(x; a_1)\, e(a_1)\, da_1$$

Finally we get that for $t \to \infty$

$$u = -2\mu \frac{\varphi_x(x; t)}{\varphi(x; t)} \sim 2\mu \frac{\int [\partial Z_{t_1}(x; a_1)/\partial x]\, e(a_1)\, da_1}{\int Z_{t_1}(x; a_1)\, e(a_1)\, da_1} \tag{8}$$

The rhs of (8) is a solution of (1) periodic in space and time and (8) gives the assertion of Theorem 1.

It is clear that the properties of smoothness of

$$u^{(0)}(x; t) = -2\mu \frac{\int [\partial Z_{t_1}(x; a_1)/\partial x]\, e(a_1)\, da_1}{\int Z_{t_1}(x; a_1)\, e(a_1)\, da_1}$$

depend on the smoothness of F. In particular, if $F(x) \in C^k(S^1)$, then $u^{(0)}(x; t) \in C^{k-1}(S^1)$ for any t and one can prove easily the convergence of $(\partial^i/\partial x^i)\, u(x; t)$, $i \leqslant k-1$, to $(\partial^i/\partial x^i)\, u^{(0)}(x; t)$. Also, the convergence in Theorem 1 is uniform in x on any compact subset of R^1. Certainly in general it cannot be uniform on the whole line, because of fluctuations of v.

3. PROOF OF THEOREM 2

Again we use the Cole–Hopf substitution, which now gives the expression for φ in the form

$$\varphi(x; t) = \int_{-\infty}^{\infty} dy\, \{\exp[-v(y)]\} \times \left\{\exp\left[\int_0^t F(w(\tau))\, db(\tau)\right]\right\} d\Pi_{(x,y)}^{(t,0)}(W) \tag{9}$$

Here $\int_0^t F(w(\tau))\, db(\tau)$ is a stochastic integral, and $B(\tau) = db(\tau)/d\tau$ is white noise. It is worthwhile to stress that $\{w(\tau)\}$ and $\{b(\tau, 0)\}$ are statistically independent Brownian motions. Therefore $\varphi(x; t)$ is random because of the randomness of b. We proceed in the same way as before. Take an integer $r = r(t)$ for which $r/t \to 1$ as $t \to \infty$ and divide the interval $(0, t)$ into r equal parts. Denote the points of the division by $t = t_0 > t_1 > \cdots > t_r = 0$, and rewrite (9) as follows:

$$\varphi(x; t) = \int \cdots \int da_1 \cdots da_r \sum_{\substack{m_1, \ldots, m_r \\ m_j \in \mathbb{Z}^1}} K_1(x, a_1 + m_1 x_0) \times K_j(a_{j-1} + m_{j-1} x_0, a_j + m_j x_0) \exp[-v(a_r + m_r x_0)] \tag{10}$$

where

$$K_j(a', a'') = \int \left\{\exp\left[\int_{t_j}^{t_{j-1}} F(w(\tau))\, db(\tau)\right]\right\} d\Pi_{a', a''}^{(t_{j-1}, t_j)}(W)$$

In the case of white noise the operators $K_j(a', a'')$ are random and statistically independent in a natural sense for different j. The periodicity of F in x implies

$$K_j(a', a'') = K_j(a' + m x_0, a'' + m x_0), \qquad m \in \mathbb{Z}^1$$

This gives again a possibility to reduce the summation $\sum_{m_1,\dots,m_r}$ to a problem concerning independent differently distributed random variables. Namely, introduce the partition functions

$$Z_1(x, a_1) = \sum_m K_1(x, a_1 + mx_0)$$

$$Z_j(a_{j-1}, a_j) = \sum_{m \in \mathbb{Z}^1} K_j(a_{j-1}, a_j + mx_0)$$

and the corresponding probabilities

$$p_1(x, a_1 + m_1 x_0) = Z_1^{-1}(x, a_1)\, K(x, a_1 + m_1 x_0)$$

$$\begin{aligned} p_j(a_{j-1} &+ m_{j-1} x_0, a_j + m_j x_0) \\ &= Z_j^{-1}(a_{j-1}, a_j)\, K_j(a_{j-1} + m_{j-1} x_0, a_j + m_j x_0) \\ &= Z_j^{-1}(a_{j-1}, a_j)\, K_j(a_{j-1}, a_j + (m_j - m_{j-1})\, x_0) \end{aligned}$$

Then (10) takes the form

$$\begin{aligned} \varphi(x; t) = \int da_1\, da_2 \cdots da_r\, & Z_1(x, a_1)\, Z_2(a_1, a_2) \cdots Z_r(a_{r-1}, a_r) \\ \times & \sum_{n_1, n_2, \dots, n_r \in \mathbb{Z}^1} p_1(x, a_1 + n_1 x_0) \\ \times & \; p_2(a_1, a_2 + n_2 x_0) \cdots p_r(a_{r-1}, a_r + n_r x_0) \\ \times & \; \exp\{-r[a_r + (n_1 + \cdots + n_r)\, x_0]\} \end{aligned}$$

Let $\xi_1, \dots, \xi_r$ be r independent integer-valued random variables where each ξ_j has the distribution $p_j(a_{j-1}, a_j + mx_0)$, $a_0 = x$. We can write

$$\begin{aligned} \sum_{n_1, n_2, \dots, n_r \in \mathbb{Z}^1} & p_1(x, a_1 + n_1 x_0)\, p_2(a_1, a_2 + n_2 x_0) \cdots p_r(a_{r-1}, a_r + n_r x_0) \\ & \times \exp\{-r[a_r + (n_1 + \cdots + n_r)\, x_0]\} \\ = & E_\xi \exp\{-r[a_r + (\xi_1 + \cdots + \xi_r)\, x_0]\} \end{aligned} \tag{11}$$

Again as in Section 2 we encounter two problems. The first one concerns the validity of the local central limit theorem of probability theory, while the second one consists of the possibility of replacing the average (11) by its mathematical expectation A. Since the distribution P_0 has the properties 1–3 (see Section 1), the second problem is simple because the local central limit theorem and the stability of the averages (see property 3) of the distribution P_0 show that (11) is equivalent to A as $t \to \infty$.

In order to study the local central limit theorem, introduce

$$\mu_1(a_1)=E\xi_1, \qquad \mu_j(a_{j-1}, a_j)=E\xi_j, \qquad d_1(a_1)=D(\xi_1)$$

$$d_j(a_{j-1}, a_j)=D(\xi_j), \qquad 2\leqslant j\leqslant r, \qquad \mathcal{M}_r=\mu_1(a_1)+\sum_{j=2}^{r} \mu_j(a_{j-1}, a_j)$$

$$D_r=d(a)+\sum_{j=2}^{r} d_j(a_{j-1}, a_j)$$

Certainly, $\mathcal{M}_r$ and D_r are andom variables, since they are functions of b.

Let $t\to\infty$. Consider the probability $P_b(t)$ (with respect to b) that the random variables $\xi_1, \xi_2,..., \xi_r$ satisfy the local central limit theorem (lclt) in the form described in the Lemma 1.

Lemma 3. $P_b(t)\to 1$ as $t\to\infty$.

The proof of the lemma is simple and we shall describe only the main steps. It uses characteristic functions. It is easy to show that there exists a finite covering of S^1 by arcs $C_0, C_1, C_2,..., C_s$, $p>0$, $\delta>0$, such that C_0 is a symmetric neighborhood of 1 and for any C_j, $1\leqslant j\leqslant s$, the probability (with respect to B) that the characteristic function has on C_j the absolute value less than $1-\delta$ is greater than p. This gives easily an exponential estimation for the characteristic function of the sum $\sum_{j-1}^{r}\xi_j$ outside a small neighborhood of 1. The rest follows the traditional way of proving the local central limit theorem.[4]

Thus, under the conditions of Theorem 1 and for those b for which the lclt is true we can write again

$$\varphi(x; t)\sim A\int da_1\, da_2\cdots da_r\, Z_1(x, a_1)\prod_{j=2}^{r} Z_j(a_{j-1}, a_j) \tag{12}$$

Now $Z_j(a_{j-1}, a_j)$ are b-independent random variables. The analysis of (12) can be done again with the help of the theory of non-homogeneous Markov chains.

Namely, consider the conditional probabilities

$$\pi_1(a_1/a_2)=\frac{Z_2(a_1, a_2)}{\int Z_2(a_1, a_2)\, da_1}$$

$$\pi_j(a_j/a_{j+1})=\frac{\int Z_2(a_1, a_2)\cdots Z_j(a_{j-1}, a_j)\, Z_{j+1}(a_j, a_{j+1})\, da_1\cdots da_{j-1}}{\int Z_2(a_1, a_2)\cdots Z_j(a_{j-1}, a_j)\, Z_{j+1}(a_j, a_{j+1})\, da_1\cdots da_j}$$

We can use them to rewrite the rhs of (12) in another way:

$$\varphi(x;t) \sim A\,\Xi_r \int Z_1(x, a_1)\,\pi_1(a_1 \mid a_2)\,\pi_2(a_2 \mid a_3)\cdots$$
$$\times\ \pi_{r-1}(a_{r-1} \mid a_r)\,\pi_r(a_r)\,da_1 \cdots da_r$$

where

$$\Xi_r = \int da_1 \cdots da_r\, Z_2(a_1, a_2) \cdots Z_r(a_{r-1}, a_r)$$

plays the role of partition function. For the derivative $\varphi_x(x;t)$ we have a similar expression:

$$\varphi_x(x;t) \sim A\Xi_r \int \frac{\partial Z_1}{\partial x}(x, a_1)\,\pi_1(a_1 \mid a_2)\,\pi_2(a_2 \mid a_3)\cdots$$
$$\times\ \pi_{r-1}(a_{r-1} \mid a_r)\,\pi(a_r)\,da_1 \cdots da_r$$

Therefore this yields for the solution $u(x;t)$ of the Burgers equation

$$u(x,t) = -2\mu\frac{\varphi_x(x;t)}{\varphi(x;t)} \sim \frac{-2\mu \int (\partial Z_1/\partial x)(x, a_1)\,\pi_1(a_1 \mid a_2)\cdots}{\int Z_1(x, a_1)\,\pi_1(a_1 \mid a_2)\cdots}$$
$$\times\ \frac{\cdots \pi_{r-1}(a_{r-1} \mid a_r)\,\pi(a_r)\,da_1 \cdots da_r}{\cdots \pi_{r-1}(a_{r-1} \mid a_r)\,\pi(a_r)\,da_1 \cdots da_r}$$
$$= \frac{\int (\partial Z_1/\partial x)(x, a_1)\,\pi_1(a_1 \mid a_k)\,\pi_k(a_{k+1} \mid a_k)\cdots}{\int Z_1(x, a_1)\,\pi_1(a_1 \mid a_k)\,\pi_k(a_{k+1} \mid a_k)\cdots}$$
$$\times\ \frac{\pi_{r-1}(a_{r-1} \mid a_r)\,\pi(a_r)\,da_1\,da_k \cdots da_r}{\pi_{r-1}(a_{r-1} \mid a_r)\,\pi(a_r)\,da_1\,da_k \cdots da_r}$$

for any k. Here $\pi(a_1 \mid a_k)$ is the conditional density corresponding to the joint probability density

$$\frac{Z_2(a_1, a_2) \cdots Z_r(a_{r-1}, a_r)\,da_1\,da_2 \cdots da_r}{\Xi}$$

Now we remark that for large k the conditional distribution $\pi(a_1 \mid a_k)$ becomes almost independent of a_k, $a_{k+1}, \ldots, a_r$ and thus independent of $B(\tau)$, $0 \leqslant \tau \leqslant t_k$. This follows easily from the ergodic theorem for Markov chains. To be more precise, let us formulate the following lemma.

Lemma 4. There exist positive constants $\rho < 1$ and $C_2 < \infty$ and events $S_k \in B(0, k)$, $k = 1, 2, \ldots$, $P_b(S_k) > 1 - C_2\rho^k$ and a functional

$H_x^{(k)}(b(t_1, t_2),\ 0 \leqslant t_1,\ t_2 \leqslant k)$ defined on $B(0, k)$ such that if $(b(t_1, t_2),\ t-k \leqslant t_1 < t_2 \leqslant t) \in S_k$, then

$$|u(x, t) - H_x^{(k)}(b(t_1, t_2), t-k \leqslant t_1 < t_2 \leqslant t)| \leqslant C_2 \rho^k$$

The functional $H_x^{(k)}$ is a periodic function of x of period x_0.

The proof of the lemma goes as follows. The transition densities $\pi_j(a_{j-1} \mid a_j)$ are bigger than some constant $\sigma > 0$ with a positive probability. It is easy to show that with the probability not less than $1 - C_2 \rho^k$ the number of such j is bigger then βk for some $\beta > 0$. Then the conditional distribution $\pi(a_1 \mid a_k)$ does not depend on k. The periodicity of $H_x^{(k)}$ on x follows easily from the expressions for $\varphi(x; t)$.

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REFERENCES

1. E. Hopf, *Commun. Pure Appl. Math.* **3**:201 (1950).
2. J. M. Burgers, *The Nonlinear Diffusion Equation* (Reidel, Dordrecht, 1974).
3. B. Simon, *Functional Integration and Quantum Physics* (Academic Press, New York, 1979).
4. B. V. Gnedenko and A. N. Kolmogorov, *Limit Theorems for Sums of Independent Random Variables* (FizMatGiz, Moscow, 1949).